

RATIONALLY CONNECTED VARIETIES OVER THE MAXIMALLY UNRAMIFIED EXTENSION OF P-ADIC FIELDS

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ABSTRACT. A result of Graber, Harris, and Starr shows that a rationally connected variety defined over the function field of a curve over the complex numbers always has a rational point. Similarly, a separably rationally connected variety over a finite field or the function field of a curve over any algebraically closed field will have a rational point. Here we show that rationally connected varieties over the maximally unramified extension of the p -adics usually, in a precise sense, have rational points. This result is in the spirit of Ax and Kochen's result saying that the p -adics are usually C_2 fields. The method of proof utilizes a construction from mathematical logic called the ultraproduct. The ultraproduct is used to lift the de Jong, Starr result in the equicharacteristic case to the mixed characteristic case.

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1. INTRODUCTION

Let X be a proper, smooth variety over a field K and \overline{K} an algebraic closure of K . A guiding principle in the study of K rational points on X is given by Kollár ([Kol96] IV.6.3):

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Principle 1. *If $\overline{X} = X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})$ is rationally connected, then X should have lots of K -points, at least if K is nice (e.g. K is a finite field, a function field of a curve, or a sufficiently large number field).*

The term “nice” has since been replaced by many with the term quasi-algebraically closed. A field K is said to be quasi-algebraically closed or C_1 if every homogeneous polynomial over K with degree less than the number of variables has a nontrivial solution in K . Some well known examples are finite fields, function fields in one variable over an algebraically closed field, complete discrete valuation rings with algebraically closed residue class field, and the maximal unramified extension of p -adic fields.

The hypersurface associated to a form with degree less than the number of variables, when smooth, is a Fano variety, thus is rationally connected (see [Cam92], [KMM92]). Since smooth Fano hypersurfaces defined over quasi-algebraically closed fields always have a K -rational point, it is natural to ask:

Question 2. ([Wit] 1.11) *Let X be a proper, smooth separably rationally connected variety over a field K where K is quasi-algebraically closed field. Is $X(K) = \emptyset$?*

Affirmative answers to this question have been given when:

- (1) K is the function field of a curve defined over an algebraically closed field of characteristic zero [GHS03].
- (2) K is the function field of a curve defined over an algebraically closed field of positive characteristic [dJS03].
- (3) K is a finite field [Esn03].
- (4) If X is a smooth, proper, rational surface over a quasi-algebraically closed field K , then $X(K) \neq \emptyset$ [Man66] , [CT87].

Colliot-Thélène and Madore have shown that there exist fields K of cohomological dimension 1 and del Pezzo surfaces X_d of degrees $d = 2, 3, 4$ such that $X_d(K) = \emptyset$ [CTM04]. In particular, these fields are examples of fields of cohomological dimension 1 which are not C_1 . This seems to rule out a cohomological proof that separably rationally connected varieties over C_1 fields have points.

Let us recall some basic facts about rational connectivity. Suppose that X is a smooth, projective variety over an uncountable algebraically closed field of characteristic zero. The following notions of being rationally connected are equivalent [Kol96].

- (1) Through any two closed points $x_1, x_2 \in X$ there is a morphism $f : \mathbb{P}^1 \rightarrow X$ such that $x_1, x_2 \in f(\mathbb{P}^1)$.

- (2) Through any number of closed points $x_0, x_1, \dots, x_n \in X$ there is a rational curve $f : \mathbb{P}^1 \rightarrow X$ passes through x_0, \dots, x_n with prescribed jet data.
- (3) There is a rational curve $f : \mathbb{P}^1 \rightarrow X$ such that f^*T_X is ample.

A curve satisfying condition 3 is called a very free rational curve. It is called this since f may be deformed along X while fixing a given point.

The proof that conditions 1-3 are equivalent relies in some way on generic smoothness, which fails in characteristic p . In fact, in positive characteristic there are smooth, projective varieties satisfying condition 1, but not having any very free curves [SK79]. Over any algebraically closed field, a smooth projective variety containing a very free curve is said to be separably rationally connected. Such varieties necessarily satisfy conditions 1 and 2.

Because of these anomalies, the Graber, Harris, Starr result does not extend to rationally connected varieties in the sense of condition 1 above. An example of a rationally connected but not separably rationally connected variety over a field of positive characteristic is give by Kollár ([Kol96] V.5.19). However, the result does extend to separably rationally connected varieties. The question we consider in this article is whether or not a smooth, projective, rationally connected variety over the maximal unramified extension of the p -adics, \mathbb{Q}_p^{nr} , has a rational point. Lang's theorem asserts that this is true for Fano hypersurfaces [Lan52]. Here we prove a partial result.

Theorem 3. *Fix an arithmetic polynomial P . There is a finite set of exceptional primes $e(P)$, depending only on P , so that if X is a smooth, projective, rationally connected variety defined over \mathbb{Q}_p^{nr} with Hilbert polynomial P , then $X(\mathbb{Q}_p^{nr}) \neq \emptyset$ as long as $p \notin e(P)$.*

It is worth noting that for any set of smooth, projective rationally connected varieties with fixed invariants there exist a finite number of primes away from which all reductions are separably rationally connected. Our proof does not show whether or not this the set of exceptional primes.

This theorem is similar to Ax and Kochen's theorem [AK65] that the p -adic number fields are almost C_2 . In fact, the main method of proof is the same. The maximally unramified extension of the p -adics are compared in a precise way to the completion of function fields of curves in characteristic p . The result of de Jong and Starr can then be lifted via the methods of Ax and Kochen.

Emil Artin conjectured that the p -adic fields \mathbb{Q}_p are C_2 . In general, a C_i field K is one for which any form in $K[x_1, \dots, x_n]_d$ with $n > d^i$ has a nontrivial zero. In [Ter66] Terjanian found a counter example to

Artin's conjecture, see for instance [Ser73]. However, using the methods of mathematical logic, Ax and Kochen were able to show that \mathbb{Q}_p is almost C_2 in the following sense.

Theorem 4. (*Ax and Kochen*) *Fix an integer $d > 0$. Then there exists a finite number of primes p_0, \dots, p_m such that for all forms $f \in \mathbb{Q}_p[x_1, \dots, x_n]_d$ with $n > d^2$ and $p \neq p_0, \dots, p_m$, f represents zero over \mathbb{Q}_p .*

Their method of proof uses mathematical logic to make precise the analogy that \mathbb{Q}_p is like $\overline{\mathbb{F}_p}((t))$. Then using the fact that the field $\overline{\mathbb{F}_p}((t))$ is C_2 [Gre66] is enough for Ax and Kochen to conclude the above theorem.

The first section of this article proves that every smooth, projective, separably rationally connected variety over $\mathbb{F}_p((t))$ has a rational point. This follows directly from de Jong and Starr [dJS03], and is known to experts. However, we provide a proof here for completeness. It should be noted that Colliot-Thélène has also given a proof of Theorem 5 [CT]. Next, we use the tools developed by Ax and Kochen to prove Theorem 3 by lifting the result over $\mathbb{F}_p((t))$ to \mathbb{Q}_p^{nr} .

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2. EQUICARACTERISTIC CASE

Theorem 5. *Let X be a smooth, projective, separably rationally connected variety over $k((t))$ where k is algebraically closed. Then X contains a $k((t))$ -rational point.*

First, notice that this proposition is equivalent to the following. A flat family $\mathfrak{X} \rightarrow \text{Spec } k[[t]]$ with generic fiber X admits a section. To see this equivalence, let H be the irreducible component of the Hilbert Scheme containing X . We have a morphism

$$\begin{array}{ccc} \text{Spec } k((t)) & \longrightarrow & H \\ \eta & \longmapsto & [X] \end{array}$$

where η is the generic point of $\text{Spec } k((t))$. Now since H is proper, the valuative criterion for properness extends this morphism uniquely to a morphism $f : \text{Spec } k[[t]] \rightarrow H$. By the universal property of the Hilbert Scheme there is a corresponding flat family $\mathfrak{X} \rightarrow \text{Spec } k[[t]]$ with generic fiber X . Now producing a section of this family and passing to the generic fiber will provide us with a $k((t))$ valued point of X .

To produce this section we will use the following two tools. The first is a theorem of Graber, Harris and Starr [GHS03], and de Jong, Starr [dJS03].

Theorem 6. *(Graber, Harris, Starr, de Jong) Let B be a smooth projective curve over k and $\pi : Z \rightarrow B$ a flat family whose generic fiber is a smooth separably rationally connected projective variety. Then π admits a section.*

The second is a result of Greenberg [Gre66] that essentially says to solve a system of equations over a complete discrete valuation ring it suffices to solve those equations modulo some sufficiently large power of the maximal ideal.

Theorem 7. *(Greenberg) Let (R, m) be a complete discrete valuation ring with perfect residue class field and maximal ideal m , and Z is a scheme of finite type over R . There exists an integer N_Z depending on Z such that for all $N \geq N_Z$ the existence of an R/m^{N+1} valued point of $Z \times \text{Spec } R/m^{N+1}$ implies the existence of an R valued point of Z .*

Thus, given a flat family $\pi : Z \rightarrow \text{Spec } R$ there is an integer N_Z such that for all $N \geq N_Z$ a section of $Z \times \text{Spec } R/m^{N+1} \rightarrow \text{Spec } R/m^{N+1}$ implies the existence of a section of π .

Using these two results we will produce a section of $\mathfrak{X} \rightarrow \text{Spec } k[[t]]$. Basically we will approximate this family by a family over a curve. For this approximate family we will produce a section using Theorem 6, and then conclude using Theorem 7.

Proof. Recall that we have a morphism $f : \text{Spec } k[[t]] \rightarrow H$ corresponding to the family $\mathfrak{X} \rightarrow \text{Spec } k[[t]]$. Consider the graph of f , $\Gamma(f) \subset \text{Spec } k[[t]] \times H \subset \mathbb{P}_k^1 \times H$. We can find a curve B smooth and tangent to $\Gamma(f)$ at $P := (\eta, [X])$ to order N .

To find such a curve blow up $\mathbb{P}_k^1 \times H$ at P . Now any curve intersecting the exceptional divisor at the point corresponding to the tangent direction of $\Gamma(f)$ at P will blow down to a curve tangent to $\Gamma(f)$ at P to at least order one. We can find such a curve by Bertini's Theorem. Generally, to find a curve with prescribed tangency data up to order N we simply blow up N times at the points on the appropriate exceptional divisor corresponding to the tangency data and then find a curve passing through the final point.

Now projecting a general such B onto H yields a morphism $g : B \rightarrow H$. By construction g agrees with f up to order N . If we let $b \in B$ be the point in B mapping to $[X]$, then g agreeing with f up to order N is

equivalent to the following diagram commuting

$$\begin{array}{ccc}
 \mathrm{Spec} \, k[[t]]/(t^{N+1}) & \longrightarrow & \mathrm{Spec} \, k[[t]] \\
 \downarrow \alpha & & \searrow f \\
 & & H \\
 & & \nearrow g \\
 \mathrm{Spec} \, O_{b,B}/m_{b,B}^{N+1} & \longrightarrow & B
 \end{array}$$

where α is an isomorphism over k .

Note that $g : B \rightarrow H$ corresponds to a flat family $\gamma : W \rightarrow B$ with $\gamma^{-1}(b) = X$. Applying the universal property of the Hilbert Scheme to this diagram we obtain

$$\begin{array}{ccc}
 \mathfrak{X} \times \mathrm{Spec} \, k[[t]]/(t^{N+1}) & \longrightarrow & W \times \mathrm{Spec} \, O_{b,B}/m_{b,B}^{N+1} \\
 \downarrow & & \downarrow \\
 \mathrm{Spec} \, k[[t]]/(t^{N+1}) & \longrightarrow & \mathrm{Spec} \, O_{b,B}/m_{b,B}^{N+1}
 \end{array}$$

where the two horizontal arrows are isomorphisms.

Now since $\gamma^{-1}(b) = X$ is a smooth separably rationally connected variety and rational connectivity is an open condition, the generic fiber of $\gamma : W \rightarrow B$ is separably rationally connected. By Theorem 6 this has a section $B \rightarrow W$. Reducing this section modulo t^{N+1} and utilizing the above diagram we obtain a section of

$$\mathfrak{X} \times \mathrm{Spec} \, k[[t]]/(t^{N+1}) \rightarrow \mathrm{Spec} \, k[[t]]/(t^{N+1})$$

Since we have no limits on how large we can choose N to be, Theorem 7 yields a section of our original family

$$\mathfrak{X} \rightarrow \mathrm{Spec} \, k[[t]]$$

In particular, $X(k((t))) \neq \emptyset$. □

3. MIXED CHARACTERISTIC CASE

Let k be a perfect field, and let $W(k)$ denote the fraction field of the Witt Vectors over k . In general, the Witt Vectors over a field k is the unique unramified complete discrete valuation ring with residue class field k (see for instance [Ser73]). Ax and Kochen's methods can be used to compare $W(\mathbb{F}_p)$ and $\mathbb{F}_p((t))$ from which it follows that $W(\mathbb{F}_p)$ has an almost C_1 property. However, there is a stronger result due to Lang showing that $W(\mathbb{F}_p)$ is actually C_1 .

In what follows we demonstrate how Ax and Kochen's theorems combined with Theorem 5 give us the theorem below.

Theorem 8. *Fix an arithmetic polynomial P . Let X be a smooth, projective, separably rationally connected variety defined over $W(\bar{\mathbb{F}}_p)$ with Hilbert polynomial P . There is a finite set of exceptional primes $e(P)$, depending only on P , so that X has a rational point as long as $p \notin e(P)$.*

It may actually be the case that such varieties over $W(\bar{\mathbb{F}}_p)$ always have rational points as in the case of Fano hypersurfaces, i.e. the set of exceptional primes, $e(P)$, is empty. It follows from a theorem of Lang that Theorem 8 implies Theorem 3 [Lan52]. The theorem says in particular that a variety over \mathbb{Q}_p^{nr} that has a point over $W(\bar{\mathbb{F}}_p)$ actually has a point in \mathbb{Q}_p^{nr} .

To proceed we have to make the comparison of $W(\bar{\mathbb{F}}_p)$ to $\mathbb{F}_p((t))$ precise. Ax and Kochen do this via the mathematical logic construction of ultrafilters and ultraproducts. After some preliminaries on these two constructions we state our goal in this precise language. A more thorough introduction to ultrafilters and ultraproducts is given in [Koc75].

Definition 9. Let S be a set and let Σ be a collection of non-empty subsets of S . Then Σ is called a *non-principal filter* if the following hold:

- (1) $S_1, S_2 \in \Sigma$ implies $S_1 \cap S_2 \in \Sigma$
- (2) $S_1 \in \Sigma$ and $S_2 \supset S_1$ implies $S_2 \in \Sigma$
- (3) For each $s \in S$ there is a set $S_1 \in \Sigma$ such that $s \notin S_1$

Σ is called a *non-principal ultrafilter* if it is maximal among the class of all non-principal filters on S , or equivalently:

- (4) $S_1 \notin \Sigma$ implies $S - S_1 \in \Sigma$.

Conditions 1, 2, 4 define an ultrafilter and 1–4 defines a non-principal ultrafilter. However, we will assume all ultrafilters to be non-principal.

A simple, but important property of ultrafilters to keep in mind is that if S is the disjoint union of subsets S_1, \dots, S_n , then precisely one of these subsets is in Σ . This observation follows from properties 1 and 3. Namely, at least one of the S_i is in Σ by property 3. Moreover, two disjoint subsets cannot both be in Σ since then so would there intersection, but Σ consists only of nonempty subsets of S .

Given any subset $S_0 \subset S$ it will be useful to know if we can find an ultrafilter on S containing S_0 . Certainly, if S_0 is a finite set, then property 4 of ultrafilters will prevent us from finding an ultrafilter containing S_0 . However, this is the only obstruction as the below lemma asserts.

Lemma 10. *Given any infinite subset $S_0 \subset S$, there exists an ultrafilter containing S_0 .*

Proof. Let Σ consist of all the subsets of S that contain all but a finite number of points in S_0 . It is easy to check that Σ is a non-principal filter on S containing S_0 . The non-principal ultrafilter desired is the maximal filter containing Σ . \square

Our usage of ultrafilters will be for an auxiliary construction called the ultraproduct. In particular, given a collection of fields indexed by a set S , and an ultrafilter Σ on S , we will construct a new field via the ultraproduct. We will use a similar construction for modules, and sheaves of modules.

Definition 11. Given an ultrafilter Σ on S and a collection of rings $\{R_i\}_{i \in S}$ we can form a new ring denoted

$$\prod_{i \in S} R_i / \Sigma$$

defined by component wise addition and multiplication under the equivalence condition that $a, b \in \prod_{i \in S} R_i$ are equivalent if they agree on a set of indices in Σ . This new ring is called the *ultraproduct* of the R_i 's with respect to Σ .

The same definition can be made for groups, modules, etc. The following lemma shows that ultraproducts on fields have nicer properties than just the product of fields:

Lemma 12. *If $\{F_i\}$ is a collection of fields indexed by a set S , and Σ is any ultrafilter on S , then the ultraproduct of the F_i 's with respect to Σ is a field.*

Proof. Let $a \in \prod_{i \in S} F_i / \Sigma$ be an element of the ultra product of the fields F_i and let a_i be a representative for a in F_i . Let S_a be the subset of S where the a_i are zero. If $S_a \in \Sigma$, then a is equivalent to the zero element. If S_a is not contained in Σ , then $S - S_a \in \Sigma$ and a is equivalent to an element b where none of the b_i are zero. The multiplicative inverse of a is then just the inverse of b . \square

Lemma 13. ([AK65] Lemma 4) *Let $S \subset \mathbb{N}$ be the set of integer primes, k_p a collection of fields of characteristic p indexed by S , and Σ an ultrafilter on S . Then*

$$\prod_{p \in S} k_p / \Sigma$$

is a field of characteristic zero.

Lemma 14. *For each $i \in S$, let M_i be a free module of rank less than N over a ring R_i . Then an ultraproduct of the M_i 's is a free module of rank less than N over the corresponding ultraproduct of the R_i 's.*

Proof. First, assume that the rank of the M_i 's are all $m > 0$. Then for any ultrafilter Σ on S

$$M := \prod_{i \in S} M_i / \Sigma$$

will be a free module of rank m over

$$R := \prod_{i \in S} R_i / \Sigma.$$

To see this, let e_{i1}, \dots, e_{im} be an R_i basis for M_i . Then note that M has basis $(e_{i1})_{i \in S}, \dots, (e_{im})_{i \in S}$ over R .

Now generally, consider the subsets $S_k \in S$ consisting of those $i \in S$ such that the rank of M_i is k . Then S is the disjoint union of S_1, \dots, S_N . By the remarks on the definition of ultrafilter, there is only one such subset contained in Σ , say $S_m \in \Sigma$. It follows that M has rank m over R . \square

A version of Ax and Kochen's theorem can be stated as:

Theorem 15. *Denote by S the set of all integer primes. Let $\{k_p\}_{p \in S}$ be a family of algebraically closed fields of characteristic p . Then for any ultrafilter Σ on S ,*

$$\prod_{p \in S} k_p((t)) / \Sigma \simeq \prod_{p \in S} W(k_p) / \Sigma$$

For a self contained proof of this see [Koc75].

To conclude Theorem 8 from this proposition we will develop some basic algebraic geometry over a general ultraproduct of fields $F = \prod_{i \in S} F_i / \Sigma$. Similar constructions can be found in papers of Arapura [Ara] and Schoutens [Sch05]. Suppose we are given a scheme X of finite type over F . There is a natural process to obtain schemes X_i of finite type over F_i , and for almost every $i \in S$, X_i is nicely related to X . However, the X_i are not unique.

Let's assume first that X is an affine scheme corresponding to the F -algebra

$$F[x_0, \dots, x_n] / I(X).$$

Suppose that f_1, \dots, f_k are generators for $I(X)$. We may write each generator as

$$f_j = \sum a_{j,I} x^I, \quad I \in \mathbb{N}^{n+1}.$$

Let $(a_{j,I}^i)_{i \in S} \in \prod_{i \in S} F_i$ be a representative for $a_{j,I}$. Setting

$$f_j^i = \sum a_{j,I}^i x^I,$$

we define X_i as the affine scheme associated to the ideal generated by the f_j^i . These schemes are not unique as they depend on the choice of representatives for the $a_{j,I}$. However, we shall see that almost every X_i is related nicely to X . In particular, under some restrictive hypothesis we have a converse. Namely, given for each $i \in S$ a scheme of finite type X_i over F_i sometimes we can lift these to a scheme of finite type over F .

Lemma 16. *Let $X \subset \mathbb{P}_F^n$ be a projective variety over F with Hilbert polynomial P . Then for almost every $i \in S$, X_i has Hilbert Polynomial P . Conversely, given a collection of projective X_i over F_i with Hilbert polynomial P for almost every $i \in S$ we can define*

$$\prod_{i \in S} X_i / \Sigma,$$

which is a projective variety in \mathbb{P}_F^n with Hilbert Polynomial P .

Proof. Let $J_i \subset F_i[x_0, \dots, x_n]$ be the homogeneous ideal of $X_i \subset \mathbb{P}_{F_i}^n$. For each degree $d > 0$ consider the F_i -vector space $J_{i,d}$ of the homogeneous polynomials of degree d in J_i .

Now define

$$J_d := \prod_{i \in S} J_{i,d} / \Sigma$$

It is a property of the Hilbert polynomial that for sufficiently large d the rank of $J_{i,d}$ is the same for each $i \in S$, i.e. the Hilbert functions of the X_i are equal for sufficiently large d . Then, the proof of Lemma 14 shows that for $d \gg 0$ the rank of J_d equals the rank of $J_{i,d}$. This yields a homogeneous ideal

$$J := \bigoplus_{d > 0} J_d \subset F[x_0, \dots, x_n].$$

The corresponding projective variety X denoted by

$$X := \prod_{i \in S} X_i / \Sigma$$

has Hilbert Polynomial P . □

Notice that X is smooth over F if and only if X_i is smooth over F_i for almost every $i \in S$. The main technical result that we need to prove Theorem 8 is formulated below.

Proposition 17. *Suppose that $X_i \subset \mathbb{P}^n$ are projective varieties defined over F_i indexed by the set S , and let X be the ultraproduct of the X_i with respect to some ultrafilter on S . Then X is smooth, separably rationally connected over F if and only if almost every X_i is smooth, separably rationally connected over F_i .*

The proof of this proposition will be given in the remainder of the paper. However, using this proposition we can now prove Theorem 8.

Proof. Suppose by way of contradiction that there is an infinite subset of primes $S_0 \subset S$ such that for each $p \in S_0$ there is an $X_p \in \mathbb{P}^n$ defined over $W(\bar{\mathbb{F}}_p)$ having Hilbert Polynomial P without a rational point. Now by Lemma 10 there is an ultrafilter Σ containing S_0 . Define the two ultraproducts

$$k = \prod_{p \in S} W(\bar{\mathbb{F}}_p) / \Sigma$$

$$k' = \prod_{p \in S} \bar{\mathbb{F}}_p((t)) / \Sigma.$$

We have by Proposition 15 that $k \simeq k'$.

Now, since each X_p for $p \in S_0$ has Hilbert Polynomial P , we get a smooth projective rationally connected variety X over k by Lemma 16 and Proposition 17. Notice that X has no rational point over k since none of the X_p for $p \in S_0 \in \Sigma$ have rational points.

Since $k \simeq k'$, X may be regarded as a variety over k' . Thus by Proposition 17 for almost every p we get a smooth separably rationally connected projective variety over $\bar{\mathbb{F}}_p((t))$ none of which have a rational point. This contradicts Theorem 5, and so the theorem follows. \square

Note that the proof of the theorem appears to require that each of the X_p be contained in some fixed projective space. However, this is not necessary since there are only finitely many projective spaces in which a variety of a fixed dimension and degree can be embedded nondegenerately.

4. MORE ON ULTRAPRODUCT OF VARIETIES

This section serves mainly to prove Proposition 17. For this proof it is convenient to develop some general results on the cohomology of the ultraproduct of sheaves.

In the setting of the previous section, X is the ultraproduct of the projective varieties X_i with a given Hilbert polynomial. Given a collection of F_i -schemes X_i and (quasi)coherent sheaves \mathcal{F}_i on X_i , we define

the (quasi)coherent sheaf $\mathcal{F} = \prod_{i \in S} \mathcal{F}_i / \Sigma$ as the pullback of $\prod_{i \in S} \mathcal{F}_i$ to $\prod_{i \in S} X_i / \Sigma$ for any (ultra)filter Σ [Ara].

This definition has certain nice properties one would hope for. Namely, if \mathcal{F} is a coherent sheaf of ideals on X , then almost all \mathcal{F}_i are sheaves of ideals, and the closed subscheme they determine on X_i is a representative of the closed subscheme determined by \mathcal{F} on X . Furthermore, being invertible or locally free holds for the sheaf \mathcal{F} if and only if they hold for almost all \mathcal{F}_i [Sch05]. The following properties also hold:

- (i) the sheaf of differentials is compatible with ultraproducts:

$$\Omega_X^1 \simeq \prod_{i \in S} \Omega_{X_i}^1 / \Sigma.$$

- (ii) if almost every X_i is smooth over F_i , then X is smooth over the ultraproduct of the F_i , and

$$T_X \simeq \prod_{i \in S} T_{X_i} / \Sigma.$$

- (iii)

$$\mathcal{O}_X(n) \simeq \prod_{i \in S} \mathcal{O}_{X_i}(n) / \Sigma.$$

There is a simple relationship between the cohomology of each component and the ultraproduct.

Lemma 18.

$$H^n(X, \prod_{i \in S} \mathcal{F}_i / \Sigma) = \prod_{i \in S} H^n(X_i, \mathcal{F}_i) / \Sigma.$$

Proof. If for almost every $i \in S$,

$$\dots \rightarrow \mathcal{F}_i^2 \rightarrow \mathcal{F}_i^1 \rightarrow \mathcal{F}_i \rightarrow 0$$

is a flasque resolution of \mathcal{F}_i , then

$$\dots \rightarrow \prod_{i \in S} \mathcal{F}_i^2 / \Sigma \rightarrow \prod_{i \in S} \mathcal{F}_i^1 / \Sigma \rightarrow \prod_{i \in S} \mathcal{F}_i / \Sigma \rightarrow 0$$

is a flasque resolution of

$$\prod_{i \in S} \mathcal{F}_i / \Sigma.$$

The ultraproduct of an exact sequence is defined component-wise, from which it follows that the resulting sequence is exact. Moreover, an ultraproduct of sheaves is flasque if and only if almost every component is flasque. \square

As noted by Arapura [Ara], the cohomology groups $H^n(X, \mathcal{F})$ may be infinite dimensional, even when the sheaves \mathcal{F}_i are coherent and the schemes are proper. So instead of making the dimension an integer, we assign a generalized dimension $h^n(X, \mathcal{F}) \in \prod \mathbb{N}/\Sigma$. When the \mathcal{F}_i 's are ideal sheaves, things become a little nicer.

Given an ideal sheaf \mathcal{I} on \mathbb{P}_K^ℓ , let $I := \bigoplus_{m \in \mathbb{N}} \Gamma(\mathcal{I}(m))$ and let $d(I)$ be the smallest integer such that I is generated by homogeneous polynomials of degree at most $d(I)$.

Lemma 19. ([BM93]). *Given d, n, ℓ, m there exists a constant C such for any field K and any ideal sheaf \mathcal{I} on \mathbb{P}_K^ℓ with $d(I) = d$, we have $h^n(\mathbb{P}_K^\ell, \mathcal{I}(m)) < C$. In particular, the regularity of \mathcal{I} is uniformly bounded by a constant depending only on $d(I)$ and ℓ .*

Because of the uniform bound of the regularity of ideal sheaves, we can state the following lemma which is of a similar nature to Lemmas 14, and 16.

Lemma 20. *Let X be the ultraproduct of $X_i \subset \mathbb{P}_{K_i}^\ell$, and let \mathcal{F}_i be ideal sheaves on X_i with a given Hilbert polynomial P . Then*

$$\prod_{i \in S} \mathcal{F}_i / \Sigma$$

is an ideal sheaf on X with Hilbert polynomial P .

Lemma 21. *Given morphisms*

$$\alpha_i : \mathbb{P}^1 \rightarrow X_i \subset \mathbb{P}^n$$

of degree d for almost every $i \in S$, these lift to a morphism

$$\alpha : \mathbb{P}^1 \rightarrow X$$

of degree d .

Proof. Each morphism α_i corresponds to an invertible sheaf \mathcal{L}_i on X_i . The fact that the image of α_i is contained in X_i is equivalent to saying $\alpha_i^*(\mathcal{I}_{X_i}) = 0$. Since the α_i are of a bounded degree, by Lemma 20 the \mathcal{L}_i lift to an invertible sheaf \mathcal{L} . Moreover, \mathcal{L} is ample as

$$H^1(\mathbb{P}^1, \mathcal{L} \otimes \mathcal{O}_{\mathbb{P}^1}(-2)) \simeq \prod_{i \in S} H^1(\mathbb{P}^1, \mathcal{L}_i \otimes \mathcal{O}_{\mathbb{P}^1}(-2)) / \Sigma = 0.$$

This ample invertible sheaf \mathcal{L} yields a morphism

$$\alpha : \mathbb{P}^1 \rightarrow \mathbb{P}^n.$$

Moreover, the image of α is contained in X by construction since

$$\alpha^*(\mathcal{I}_X) = \prod_{i \in S} \alpha_i^*(\mathcal{I}_{X_i}) / \Sigma.$$

□

To prove Proposition 17 we will need the following two lemmas.

Lemma 22. *Let K_i be algebraically closed fields for each $i \in S$. Fix an arithmetic polynomial P , and let X_i be smooth projective separably rationally connected varieties over K_i with Hilbert Polynomial P . Then there exists an integer $D > 0$ such that for any $i \in S$ there is a very free curve $\mathbb{P}^1 \rightarrow X_i$ of degree $\leq D$.*

Proof. Let H be the component of the Hilbert scheme parameterizing closed subschemes of \mathbb{P}^n with Hilbert polynomial P , and let $U \subset H$ be the open subscheme parameterizing smooth, separably rationally connected varieties. Note that as H is proper, U is quasi-compact.

Consider the incidence correspondence

$$Z_e = \{([X], f) \mid f \text{ is a very free curve of degree } e\} \subset U \times \text{Mor}(\mathbb{P}^1, \mathbb{P}^n)_e$$

The projection $\pi_e : Z_e \rightarrow U$ is a smooth morphism. To see this we apply the formal criterion of smoothness. Given any such diagram as below

$$\begin{array}{ccc} \text{Spec } A & \longrightarrow & Z_e \\ \downarrow & \nearrow & \downarrow \\ \text{Spec } \hat{A} & \longrightarrow & U \end{array}$$

with A an Artin local sheaf, we need to find a morphism extending the diagonal. Utilizing the universal property of the Hilbert scheme this can be translated as follows. Given a deformation of X with $[X] \in U$, find a compatible deformation of f . Since f is very free, the obstruction for doing this vanishes.

Now, since π_e is a smooth morphism it is an open mapping, and we denote its image by U_e . Moreover,

$$U = \bigcup_{e>0} U_e,$$

and so by quasi-compactness there is some integer $D > 0$ so that

$$U = U_1 \cup \dots \cup U_D.$$

□

Lemma 23. *Let $F = \prod_{i \in S} F_i / \Sigma$ be an ultraproduct field. Then $\bar{F} \subset \prod_{i \in S} \bar{F}_i / \Sigma$*

Now we are ready to prove Proposition 17.

Proof. Proving that X is separably rationally connected is equivalent to proving the existence of a very free rational curve in $X \times_F \text{Spec}(\overline{F})$ or even in $\tilde{X} := X \times_F \text{Spec}(\prod_{i \in S} \bar{F}_i/\Sigma) \simeq \prod_{i \in S} \bar{X}_i/\Sigma$.

By Lemma 22 there is an integer $D > 0$ and very free curves

$$f_i : \mathbb{P}^1 \rightarrow \bar{X}_i$$

of degree less than D . Now by Lemma 21 these curves lift to a curve

$$f : \mathbb{P}^1 \rightarrow \tilde{X}.$$

Applying what we know about the ultraproducts of sheaves we see that

$$H^1(\tilde{X}, f^*T_{\tilde{X}} \otimes \mathcal{O}_{\mathbb{P}^1}(-2)) \simeq \prod_{i \in S} H^1(\bar{X}_i, f_i^*T_{\bar{X}_i} \otimes \mathcal{O}_{\mathbb{P}^1}(-2))/\Sigma.$$

Thus f is very free if and only if almost every f_i is very free. \square

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